I U P U I

Math Club Teaser #15

March 6, 2009
(due March 13, 2009)

Solution

There are three possible outcomes from the first shot:

a) The Ugly kills the Bad: Now is the turn of the Good, who shoots the Ugly. The probability of the Ugly surviving is 0.

b) The Ugly kills the Good: The Ugly faces now a duel with the Bad, and the Bad has the first shot. The probability that the Bad misses is $\frac{1}{3}$, but the Ugly may miss also, giving the Bad one more chance. This means that the probability of the Ugly surviving is less than $\frac{1}{3}$.

c) The Ugly misses: Now is the turn of the Bad, who is certain to aim at the Good (otherwise the Good survives for sure and will take first aim at the Bad).
   - If the Bad misses (probability $\frac{1}{3}$), the Good shoots the Bad, and the Ugly has $\frac{1}{3}$ probability of winning against the Good.
   - If the Bad kills the Good (probability $\frac{2}{3}$), the Ugly faces a duel with the Bad, but now the Ugly has the first shot. The probability that the Ugly wins this duel is

     $$\left(\tfrac{1}{3}\right) + \left(\tfrac{2}{3} \cdot \tfrac{1}{3} \cdot \tfrac{1}{3}\right) + \left(\tfrac{2}{3} \cdot \tfrac{1}{3} \cdot \tfrac{2}{3} \cdot \tfrac{1}{3} \cdot \tfrac{1}{3}\right) + \dots,$$

     where the n^{th} term is the probability that the Ugly gets an n^{th} shot and wins with it. This sum is a geometric series equal to $\frac{3}{7}$. Overall, the probability of the Ugly surviving is $\frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{3}{7} = \frac{25}{63} > \frac{1}{3}$.

Comparing the three cases, the Ugly should miss on purpose for a winning probability of $\frac{25}{63} \approx 0.3968$. The Good must survive a shot from the Bad and a shot from the Ugly, meaning a winning probability of $\frac{2}{9} \approx 0.2222$. Then the Bad has a winning probability of $1 - (\frac{25}{63} + \frac{2}{9}) = \frac{8}{21} \approx 0.3809$.

Solved by:

`The Residues.`