I U P U I
Math Club Teaser #27

September 18, 2009
(due September 25, 2009)

Solution

The answer is 4562.

There is a simple formula for the number of ways of giving change using \$5, \$2, and \$1 bills. For n dollars, the number of possibilities is

$$\left[(n+4)^2/20\right],$$

where $[\cdot]$ means "nearest integer".

To learn more about this formula, and other mathematical tricks, drop by room SL-008 every Wednesday from 3:00 to 4:00 pm.

When breaking 100 dollars, the \$5, \$2, and \$1 bills must add up to a multiple of 10, say $10m$ (m can be anything from 0 to 10), and then the \$50, \$20, and \$10 bills must add up to $100-10m$. This last problem is equivalent to breaking $10-m$ dollars into \$5, \$2, and \$1 bills, because everything is just multiplied by 10.

For a given m, there are $\left[(10m+4)^2/20\right]$ ways of breaking $10m$ dollars with \$5, \$2, and \$1 bills, and $\left[((10-m)+4)^2/20\right]$ ways of breaking $100-10m$ with \$50, \$20, and \$10 bills. All combinations of these are valid, so we need to multiply them together, and sum the results for all values of m from 0 to 10:

$$\left[\tfrac{4^2}{20}\right]\cdot\left[\tfrac{14^2}{20}\right]+\left[\tfrac{14^2}{20}\right]\cdot\left[\tfrac{13^2}{20}\right]+\left[\tfrac{24^2}{20}\right]\cdot\left[\tfrac{12^2}{20}\right]+\left[\tfrac{34^2}{20}\right]\cdot\left[\tfrac{11^2}{20}\right]+\left[\tfrac{44^2}{20}\right]\cdot\left[\tfrac{10^2}{20}\right]+$$
$$\left[\tfrac{54^2}{20}\right]\cdot\left[\tfrac{9^2}{20}\right]+\left[\tfrac{64^2}{20}\right]\cdot\left[\tfrac{8^2}{20}\right]+\left[\tfrac{74^2}{20}\right]\cdot\left[\tfrac{7^2}{20}\right]+\left[\tfrac{84^2}{20}\right]\cdot\left[\tfrac{6^2}{20}\right]+\left[\tfrac{94^2}{20}\right]\cdot\left[\tfrac{5^2}{20}\right]+\left[\tfrac{104^2}{20}\right]\cdot\left[\tfrac{4^2}{20}\right]=$$

$$1\cdot10+10\cdot8+29\cdot7+58\cdot6+97\cdot5+146\cdot4+205\cdot3+274\cdot2+353\cdot2+442\cdot1+541\cdot1=\mathbf{4562}.$$

Solved by:

`Captain Nemo, The Cauchy Convergers, The Residues, John Taggart.`