Continuous Variables

CS B553
Spring 2013

Discrete random variables
• So far we’ve encoded our CPDs and potential functions as tables of discrete values
• Advantages
  – Very general – can represent any arbitrary CPD or potential function
  – Easy to understand – can always fall back on exhaustive search to find the correct answer
• Disadvantages
  – Can’t solve for continuous (real-valued) random variables
  – Can hide the structure of a distribution, making inference more difficult than necessary
  – State space can become enormous!

Graphical models with continuous distributions
• Using continuous distributions means that we’ll need to define the potential functions using equations
  – i.e. using parametric distributions: Gaussian, binomial, Dirichlet, beta, von Mises, poisson, power law...
• Most of the techniques we’ve seen so far in the class can be applied to continuous distributions
  – With one notable exception...
  – But they can (often) make inference much more difficult

Gaussian (Normal) distributions
• Let $X = \#$ of sixes when $N$ dice are rolled

$$P(X) = \mathcal{N}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left( -\frac{1}{2} \frac{(x - \mu)^2}{\sigma^2} \right)$$
$$Z = \sqrt{2\pi\sigma^2}$$

• Very popular for two main reasons:
  – Math is very convenient
  – Many distributions that arise in nature are Gaussian (why?)
• Let $X = \#$ of sixes when $N$ dice are rolled

\[ P(X = r) = \binom{N}{r} \left( \frac{1}{6} \right)^r \left( \frac{5}{6} \right)^{N-r} \]

\[
E[X] = \frac{N}{6}
\]
\[
\sigma[X] = \sqrt{\frac{N}{6}}
\]

- $N = 2$
  - $E[X] = \frac{2}{6} = 0.333$
  - $\sigma[X] = \frac{\sqrt{2}}{\sqrt{6}} = 0.527$

- $N = 3$
  - $E[X] = \frac{3}{6} = 0.500$
  - $\sigma[X] = \frac{\sqrt{3}}{\sqrt{6}} = 0.745$

- $N = 5$
  - $E[X] = \frac{5}{6} \approx 0.833$
  - $\sigma[X] = \frac{\sqrt{5}}{\sqrt{6}} \approx 1.179$

- $N = 10$
  - $E[X] = \frac{10}{6} \approx 1.667$
  - $\sigma[X] = \frac{\sqrt{10}}{\sqrt{6}} \approx 1.745$

- $N = 40$
  - $E[X] = \frac{40}{6} \approx 6.667$
  - $\sigma[X] = \frac{\sqrt{40}}{\sqrt{6}} \approx 4.714$

- $N = 80$
  - $E[X] = \frac{80}{6} \approx 13.333$
  - $\sigma[X] = \frac{\sqrt{80}}{\sqrt{6}} \approx 7.814$

- $N = 160$
  - $E[X] = \frac{160}{6} \approx 26.667$
  - $\sigma[X] = \frac{\sqrt{160}}{\sqrt{6}} \approx 12.897$

- $N = 400$
  - $E[X] = \frac{400}{6} \approx 66.667$
  - $\sigma[X] = \frac{\sqrt{400}}{\sqrt{6}} \approx 15.811$
• Let $X = \#$ of sixes when $N$ dice are rolled

$P(X = r) = \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{320-r}$

$E[X] = \frac{120}{5} = 320 \times \frac{1}{5}$

$\sigma = \sqrt{\frac{120 \times 5}{6 \times (6-1)}} = 6.67$

$S[X] = \frac{120}{5} \times \frac{1}{5} \approx 6.667$

$P(X = r) = \frac{1}{0.667^r} \exp\left(\frac{r - 53.333}{2 \times (6.667)^2}\right)$

An example

• In a certain election, the results of voting and the results of exit polls seem not to match:

<table>
<thead>
<tr>
<th>Voting results</th>
<th>Exit poll results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Candidate A</td>
<td>51.061% (958)</td>
</tr>
<tr>
<td>Candidate B</td>
<td>48.939% (1042)</td>
</tr>
</tbody>
</table>

What is the probability of this happening by chance alone?

Resulting distribution

Gaussian distribution

Resulting distribution
Multivariate Gaussian distributions

- Generalization to n dimensional spaces

\[
P(x) = \mathcal{N}(x; \mu, \Sigma) = \frac{1}{Z} \exp \left( -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right)
\]

\[
Z = \sqrt{(2\pi)^n |\Sigma|}
\]

**Important properties of covariance matrices**

- Must not be singular
- Must be positive definite
- Easy to check if two variables are independent

\[
P(x) = \mathcal{N}(x; \mu, \Sigma) = \frac{1}{Z} \exp \left( -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right)
\]

\[
Z = \sqrt{(2\pi)^n |\Sigma|}
\]

Multivariate Gaussian distributions

\[
P(x) = \mathcal{N}(x; \mu, \Sigma) = \frac{1}{Z} \exp \left( -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right)
\]

\[
Z = \sqrt{(2\pi)^n |\Sigma|}
\]

\[
\Sigma = \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}
\]

\[
\Sigma = \begin{bmatrix}
1.1420 & -0.2857 \\
-0.2857 & 0.5714
\end{bmatrix}
\]

**Important properties of multivariate normal distributions**

- Marginals of a normal distribution are ... ?
- Conditional distributions of a normal distribution are ... ?